Nonlinear Analysis: Hybrid Systems 4 (2010) 484–495

Contents lists available at ScienceDirect

Nonlinear Analysis: Hybrid Systems

journal homepage: www.elsevier.com/locate/nahs

Disturbance rejection in robust PdE-based MRAC laws for uncertain heterogeneous multiagent networks under boundary reference

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ARTICLE INFO

Article history:
Received 9 February 2009
Accepted 20 November 2009

Keywords:
Model reference adaptive control
Disturbance rejection
Parameter uncertainty
Decentralized control

ABSTRACT

Disturbance is a pervasive source of uncertainty in most applications. This paper presents model reference adaptive control (MRAC) laws for uncertain multiagent networks with a disturbance rejection capability. The algorithms proposed can also be viewed as the extension of the robust model reference adaptive control (MRAC) laws with disturbance rejection recently derived for systems described by parabolic and hyperbolic partial differential equations (PDEs) with spatially-varying parameters under distributed sensing and actuation to heterogeneous multiagent networks characterized by parameter uncertainty. The latter extension is carried out using partial difference equations (PdEs) on graphs that preserve parabolic and hyperbolic like cumulative network behavior. Unlike in the PDE case, only boundary input is specified for the reference model. The algorithms proposed directly incorporate this boundary reference input into the reference PdE to generate the distributed admissible reference evolution profile followed by the agents. The agent evolution thus depends only on the interaction with the adjacent agents, making the system fully decentralized. Numerical examples are presented as well. The resulting PdE MRAC laws inherit the robust linear structure of their PDE counterparts.

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1. Introduction

Disturbance is a common source of uncertainty for systems performing tasks in natural environments such as underwater sensor arrays. Hence, any controller design for such environments must address the issue of disturbance rejection. The extent to which a controller can reject disturbance depends strongly on the available information. In this work we presume that the disturbance is a weighted sum of known time variations, where the weights are not known. This framework can be used to decrease the effect of periodic disturbances of known frequency, by approximating the disturbance with a truncated Fourier series.

Control designs for systems represented by partial differential equations (PDEs) can be applied to multiagent networks described by partial difference equations (PdEs) on graphs by properly identifying elements of the theory of differential operators on graphs with their analogs in the spatially continuous setting. This perspective, developed in [1–3] and the references therein, permits viewing the collective behavior of coordinated agents in a network as the qualitative analog of the solution behavior of a PDE with differential operator structure matching, in some sense, that of the multiagent network dynamics. If the structure of such a PDE admits specifying some desirable properties of its spatiotemporal evolution [13–15], then forcing the multiagent network to track this evolution can be, in turn, specified as the control objective. This

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doi:10.1016/j.nahs.2009.11.005

viewpoint has been used in [4] to develop decentralized control schemes for the consensus problem based on damped wave equation. In the present paper we extend the robust model reference adaptive control (MRAC) technique for spatially-varying PDEs under distributed sensing and actuation, capable of rejecting disturbances, recently introduced in [5] – an approach based on the use of local information only – to the decentralized adaptive control of heterogeneous multiagent networks comprised by agents with parameter uncertainty. A similar extension of robust model reference adaptive control (MRAC) technique for PDEs without external disturbances introduced in [6] was presented in [7]. The control setting obtained in this work is capable of addressing nontrivial collective behavior tasks [8] for multiagent networks, such as for example, mobile decentralized autonomous underwater sensor arrays. In the latter context, the setting proposed admits forming a decentralized reference evolution profile [9] for a network consisting of agents with not necessarily equal masses and forcing this network to track this profile under slow changes of individual agents' masses over time due to fuel depletion. The desired network evolution profiles are directly incorporated into the MRAC algorithm structure. In addition, the MRAC laws presented here ensure tracking in the presence of disturbances of known time variation multiplied by unknown constants. Numerical examples are given to demonstrate the performance of the algorithms proposed.

As in [7], in the agent topology considered, some of the agents are assumed to have access to the desired trajectories through, for example, specialized sensing and/or communication tools, whereas the motion of the rest of the agents is generated solely through communication with their neighbors. The relevant structures of [5] are determined by the control objective. The resulting topologies are relevant in problems, such as mobile sensor arrays, where a large number of agents is required to navigate, but it is expensive to equip each of the agents with complex sensing and communication tools.

2. Preliminaries

Following [1,3], a multiagent network with a certain communication structure will be represented as an undirected unweighted graph $G = \{\mathsf{N}, \mathsf{E}\}$, where $\mathsf{N} = \{1, \ldots, N\}$ is the set of nodes, with each node corresponding to an individual agent, and $\mathsf{E} \subseteq \mathsf{N} \times \mathsf{N}$ is the set of edges, with each edge corresponding to a communication link between two agents. We will identify a node in N with the corresponding agent's index and denote the latter by x, so that $x \in \mathsf{N}$. The undirectedness of such a graph indicates that each of any two agents linked by an edge can access the other's state as well as its own. A graph G is said to be *connected* if every node in N can be reached from every other node in N along a sequence of edges in E.

Functions on graphs and operators on such functions are defined as follows [1,3]. Given a vector-valued function $f : \mathsf{N} \mapsto \mathbb{R}^n$, its integral over G is defined as

$$\int_G f = \sum_{x \in \mathsf{N}} f(x).$$

$L^2(G)$ is defined as the Hilbert space comprising such functions, with the inner product defined by

$$(f, g) = \int_G f^T g,$$

where f^T represents the transpose of the column vector $f \in \mathbb{R}^n$, and the corresponding norm defined by

$$\|f\| = \sqrt{\left(\int_G f^T f\right)}.$$

Partial differentiation on G is defined as

$$\partial_y f(x) = f(y) - f(x), \qquad \partial_y^2 f(x) = \partial_y f(y) - \partial_y f(x) = -\partial_y f(x)$$

for $x, y \in \mathsf{N}$. The Laplacian on G is, therefore, given by

$$\Delta f(x) = -\sum_{y \sim x} \partial_y^2 f(x) = \sum_{y \sim x} \partial_y f(x),$$

where $y \sim x$ means that nodes x and y are connected by an edge. We will make use of the properties of the Laplacian stated in Lemmas 1 and 2 (see [7] for proof).

Lemma 1. *The operator $-\Delta$ is nonnegative, i.e.,*

$$(-\Delta f, f) \geq 0,$$

for all $f \in L^2(G)$ [10].

Lemma 2. *For $f : \mathsf{N} \times \mathbb{R}^+ \mapsto \mathbb{R}^n$, i.e., $f(\cdot, t) \in L^2(G)$ for all $t \in \mathbb{R}^+$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Delta f, f) = 2(\Delta f, \dot{f}) = 2(\Delta \dot{f}, f),$$

where $\dot{f}(x, t)$ denotes the time derivative of $f(x, t)$.

Depending on the context, f is used throughout the paper to denote generic function, force, or control input.

Throughout this paper Hadamard (element-wise) product and division of vectors u and v are represented as $u \circ v$ and $\frac{[u]}{[v]}$ respectively. A scalar k operating with a vector is represented as $[k]$, where $[k]$ is a vector having all entries as k. The dimension of $[k]$ is assumed to be such that element-wise operations make sense.

3. MRAC of heterogeneous networks under agent parameter uncertainty in the presence of disturbance

In [7], a detailed discussion of the physical network, the plant, and reference models for single and double integrator networks was presented. To make this paper comprehensive, Sections 3.2.1 and 3.2.2 present a similar discussion, but in the presence of the disturbance term.

3.1. Physical network

In designing control laws for the multiagent network associated with a graph $G = \{\mathsf{N}, \mathsf{E}\}$ in terms of $\mathbb{R}^n$-valued functions on N, the space $\mathbb{R}^n$ may be invoked to represent agents positions or velocities. We focus on networks of agents moving physically in n-dimensional space, with the fixed communication structure specified by $\mathsf{E} = \mathsf{E}_o$, and the dynamics of the agent corresponding to each node x modeled by the equations

$$\begin{aligned} \dot{p}(x,t) &= v(x,t), \\ m(x)\dot{v}(x,t) &= f(x,t), \end{aligned} \tag{1}$$

where the n-vectors p, v and f represent position, velocity, and force, respectively, and scalar $m(x)$ represents mass. The network may be heterogeneous in that $m(x) \neq m(y)$ for some $x, y \in \mathsf{N}$, and the masses of all agents are assumed to be unknown. The embedding of (1) into MRAC framework is carried out through specifying plant models relevant to the agents' collective behavior tasks, while retaining the network communication structure. For certain tasks – such as coordinated maneuvering – an agent might be required to track the reference velocity trajectory, calling for the single integrator model corresponding to the second line in (1). For other tasks – such as data gathering or network relocation [11] – tracking of the reference position trajectory is of interest, calling for the use of the double integrator model corresponding to the full system (1). These two cases, in the presence of external disturbances, are treated in Sections 4.1 and 4.2 respectively.

3.2. Plant and reference models

3.2.1. Plant models

As presented in [7], let $u(x,t) : \mathsf{N} \times \mathbb{R}^+ \mapsto \mathbb{R}^n$ be a function of two independent variables and represent the plant model variable of interest. Focusing first on the agent velocity as the primary object of control, we associate the plant model variable $u \in \mathbb{R}^n$ with the agent model variable v in (1), and introduce the equation

$$\dot{u}(x,t) = a(x)f(x,t) + a(x)\sum_{i=1}^{N}\psi_i(x)\phi_i(x,t), \quad u(x,0) = \tilde{u}(x), \tag{2}$$

that mimics the second line of Eq. (1) while including a disturbance term. Eq. (2) defines a continuous time PdE on graph with initial conditions $\tilde{u}(x)$, where $f(x,t)$ is the control input and $a(x)$ is an unknown positive constant for each $x \in \mathsf{N}$. $\sum_{i=1}^{N}\psi_i(x)\phi_i(x,t)$ is a disturbance with known time variations $\phi_i(x,t)$, but unknown $\psi_i(x)$. To complete the plant model, we endow (2) with the communication structure E_o of the original network. The resulting representation, owing to its single time derivative form, will be henceforth referred to as the *single integrator network*. In analogy to spatiotemporal evolution of a PDE represented by $u(x,t)$ when x is a continuous variable, we will refer to $u(x,t)$ introduced above as representing a *spatiotemporal evolution* of a PdE.

Similarly, focusing on the agent position as the primary object of control and associating the plant model variable u with the agent model variable p in (1), we introduce the *double integrator network*

$$\ddot{u}(x,t) = a(x)f(x,t) + a(x)\sum_{i=1}^{N}\psi_i(x)\phi_i(x,t), \qquad u(x,0) = \tilde{u}_1(x), \qquad \dot{u}(x,0) = \tilde{u}_2(x) \tag{3}$$

where $f(x,t)$ is the control input, $a(x)$ is an unknown positive constant with known lower bound $A \leq a(x)$ for each $x \in \mathsf{N}$ and $\sum_{i=1}^{N}\psi_i(x)\phi_i(x,t)$ is a disturbance with known time variations $\phi_i(x,t)$ but unknown $\psi_i(x)$.

The orders of time derivatives in the plant models (2) and (3) as well as decentralized (local) sensing and actuation in both of them are now seen to closely resemble those of the parabolic and hyperbolic PDEs, respectively, under distributed sensing and actuation. This suggests that MRAC laws for uncertain parabolic and hyperbolic PDEs, evolving under the influence of distributed disturbance, introduced in [5], if properly extended to graphs, could be applied to (2) and (3). This extension is accomplished in Sections 4.1 and 4.2 with the algorithms developed paralleling the MRAC laws of [5].

3.2.2. Reference models, motion assignment, and reference inputs

MRAC laws typically assume that once a *reference model* is selected, a *reference input* is specified that generates the desired reference model behavior. In the context of plant models (2) *and* (3) the latter can be thought of as the behavior of the *reference agents*. In the present work, instead, the desired spatiotemporal evolution is specified, and only for part of the agents. The latter agents are further referred to as the *leading reference agents*, while the rest—as the *trailing reference agents*. The latter, then, are governed by the interplay between the reference model structure and the leading reference agents evolution. The motion of the latter agents is, then, generated through the interplay of the leading reference agents evolution, the network communication structure, and the reference model dynamics. These features give rise to an unconventional reference generation structure. The assigned evolution profiles are assumed throughout this work to be sufficiently smooth to guarantee continuity of all the functions involved.

With this setting at hand, the embedding of plant models (2) and (3) into MRAC laws of [5] is carried out by specifying the reference models in the form of PdEs that represent the desired network behavior, endowing these models with the communication structure of the corresponding plant models, selecting the leading reference agents and assigning the desired evolution to them to address specific tasks, and calculating the corresponding reference inputs that induce their assigned evolution. These reference inputs are then applied to the reference models, yielding the reference models with the explicit leading agents motion assignment. The goal of a MRAC law is to force the plant to follow the resulting reference model evolution under plant parameter uncertainty, while shaping the closed-loop dynamics to conform to those of the reference model. Let $w(x,t) : \mathsf{N} \times \mathbb{R}^+ \mapsto \mathbb{R}^n$ represent the reference model variable of interest. The physical units in which variable $w \in \mathbb{R}^n$ is expressed depend on the application of interest and are identical to those of the corresponding plant model variable $u \in \mathbb{R}^n$.

The reference model for the single integrator network is given by

$$\dot{w}(x,t) = \Delta w + r(x,t) \tag{4}$$

where $r(x,t)$ is expressed as

$$\begin{aligned} &r(x,t) = \dot{w} - \Delta w \quad \text{for the leading nodes;} \\ &r(x,t) = 0 \quad \text{for the trailing nodes.} \end{aligned} \tag{5}$$

The reference model for the double integrator network is given by

$$\ddot{w} = \Delta w - \dot{w} + r \tag{6}$$

where $r(x,t)$ is expressed as

$$\begin{aligned} &r(x,t) = \ddot{w} + \dot{w} - \Delta w \quad \text{for the leading nodes;} \\ &r(x,t) = 0 \quad \text{for the trailing nodes.} \end{aligned} \tag{7}$$

The details of the reference model development are given in [7].

4. MRAC law design

Once $w(x,t)$ and $r(x,t)$ are available, the development of the MRAC laws can proceed in a conventional manner by assuming that the reference models (4) and (6) are given along with the particular reference inputs that induce the desired reference velocity and position profiles, respectively. We define

$$\begin{aligned} &e(x,t) = w(x,t) - u(x,t), \qquad \eta_w^*(x) = \frac{1-a(x)}{a(x)}, \qquad \eta_r^*(x) = \frac{1}{a(x)}, \\ &\xi_w(x,t) = \eta_w(x,t) - \eta_w^*(x), \qquad \xi_r(x,t) = \eta_r(x,t) - \eta_r^*(x) \end{aligned} \tag{8}$$

where $\eta_w(x,t)$ and $\eta_r(x,t)$ are to be used as tunable control parameters; here $e(x,t)$ denotes the state error and $\xi_w(x,t)$ and $\xi_r(x,t)$ denote the parameter errors. We also note that $\dot{\eta}_w = \dot{\xi}_w(x,t)$ and $\dot{\eta}_r = \dot{\xi}_r(x,t)$.

4.1. Disturbance rejection in the single integrator networks

We now seek to design the control input f in (2), along with the adaptation law, such that $u(x,t)$ of (2) will follow $w(x,t)$ of (4) for a given $r(x,t)$ in spite of uncertainty in $a(x)$ and in the presence of a single external disturbance of known time variation multiplied by an unknown constant. The proof is similar in the case of disturbances that can be represented as unknown-weighted sum of known functions of time.

Proposition 1. *Assume that r and Δw in the reference model (4) and $\phi(x,t)$ (known time variation of disturbance) in (2) are continuous and bounded for all t. Following definition (8), for the network (2) and the reference model (4) set the control law to be*

$$f = \Delta u + g - \hat{\psi} \circ \phi \tag{9}$$

with adaptive part

$$g = ke + \eta_w \circ \Delta w + \eta_r \circ r, \tag{10}$$

where k is a positive constant, the coefficients η_w and η_r are adapted according to

$$\dot{\eta}_w = \varepsilon_w e \circ \Delta w, \qquad \dot{\eta}_r = \varepsilon_r e \circ r, \tag{11}$$

where ε_w and ε_r are positive constants. $\hat{\psi}$ is an observer for ψ updated according to

$$\dot{\hat{\psi}} = k_d e \circ \phi, \tag{12}$$

where k_d is a strictly negative constant. Then, $\|e\|$ tends to zero as $t \to \infty$.

Proof. For the disturbance observer

$$\dot{\tilde{\psi}} = -\dot{\hat{\psi}} = -k_d e \circ \phi, \tag{13}$$

where $\tilde{\psi} = \psi(x) - \hat{\psi}(x)$. Also note that

$$\dot{\xi}_w = \dot{\eta}_w = \varepsilon_w e \circ \Delta w, \qquad \dot{\xi}_r = \dot{\eta}_r = \varepsilon_r e \circ r. \tag{14}$$

The error $e(x, t)$ evolves according to

$$\begin{aligned}
\dot{e} &= \dot{w} - \dot{u} = \Delta w + r - \left(a \circ \Delta u + a \circ g - a \circ \hat{\psi} \circ \phi + a \circ \psi \circ \phi\right) \\
&= \Delta w - a \circ \Delta w + a \circ \Delta w - a \circ \Delta u + r - a \circ (ke + \eta_w \circ \Delta w + \eta_r \circ r) - a \circ \tilde{\psi} \circ \phi \\
&= a \circ \left(\frac{[1-a]}{[a]}\right) \circ \Delta w - a \circ \eta_w \circ \Delta w + a \circ \left(\frac{[1]}{[a]} - \eta_r\right) \circ r + a \circ \Delta e - ka \circ e - a \circ \tilde{\psi} \circ \phi \\
&= a \circ \left(\eta_w^* - \eta_w\right) \circ \Delta w + a \circ \left(\eta_r^* - \eta_r\right) \circ r + a \circ \Delta e - ka \circ e - a \circ \tilde{\psi} \circ \phi \\
&= a \circ \left(\Delta e - ke - \xi_w \circ \Delta w - \xi_r \circ r - \tilde{\psi} \circ \phi\right).
\end{aligned} \tag{15}$$

Consider the Lyapunov function

$$V = \frac{1}{2}\left(\begin{bmatrix} \dfrac{[e]}{[a]} \\ -\tilde{\psi}/k_d \end{bmatrix}, \begin{bmatrix} e \\ \tilde{\psi} \end{bmatrix}\right) + \frac{1}{2\varepsilon_w}(\xi_w, \xi_w) + \frac{1}{2\varepsilon_r}(\xi_r, \xi_r) \tag{16}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(G)$. Clearly,

$$\dot{V} = \left(\begin{bmatrix} \dfrac{[\dot{e}]}{[a]} \\ -\dot{\tilde{\psi}}/k_d \end{bmatrix}, \begin{bmatrix} e \\ \tilde{\psi} \end{bmatrix}\right) + \frac{1}{\varepsilon_w}(\dot{\xi}_w, \xi_w) + \frac{1}{\varepsilon_r}(\dot{\xi}_r, \xi_r),$$

and it follows from (8), (11), (13) and (15) that

$$\dot{V} = \left(\begin{bmatrix} \Delta e - ke - \tilde{\psi} \circ \phi \\ e \circ \phi \end{bmatrix}, \begin{bmatrix} e \\ \tilde{\psi} \end{bmatrix}\right) = (\Delta e, e) - k(e, e). \tag{17}$$

From Lemma 1 it follows that $(-\Delta e, e) \geq 0$, so that $-(\Delta e, e) \geq 0$ and $(\Delta e, e) \leq 0$. Since $(e, e) \geq 0$ and k is positive, $k(e, e) - (\Delta e, e) = -\dot{V}$ is positive semi-definite. Further on, due to the assumption of Proposition 1, the error system (13)–(15) is a linear system with bounded time-varying coefficients. Moreover the Lyapunov function V and its derivative $\dot{V}$ are time invariant. Hence all the conditions of Lasalle/Yoshizawa theorem [12, Thm. 8.4, p. 323] which applies to non-autonomous systems (as opposed to the classical Lasalle invariance principle) are observed to be satisfied globally and therefore $\|e\|$ tends to zero as $t \to \infty$. □

Remark 1. Boundedness of r does not limit the range of closed-loop system velocities, since, as presented in [7], r is computed though the derivatives of the assigned velocity profiles. Boundedness of Δw simply precludes assignment of velocity profiles in $\mathbb{R}^n$ that result in unbounded separation between agents. □

Note that the above proof works in the absence of the Laplacian. But the Laplacian is not introduced here for the sake of convergence. It is introduced in the reference model so that given reference inputs for a few agents, the remaining agents can move in a fashion dictated by the Laplacian.

Combining the results of this subsection with the single integrator network results of Section 3.2 yields MRAC schematics shown in Fig. 1.

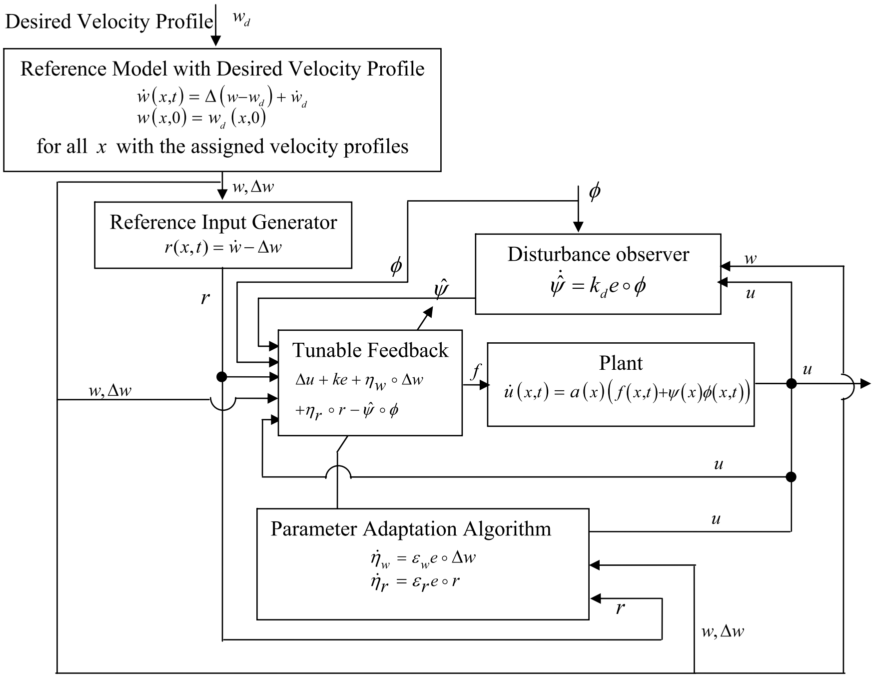

Fig. 1. Implementation schematics of the MRAC law with disturbance rejection for the single integrator network.

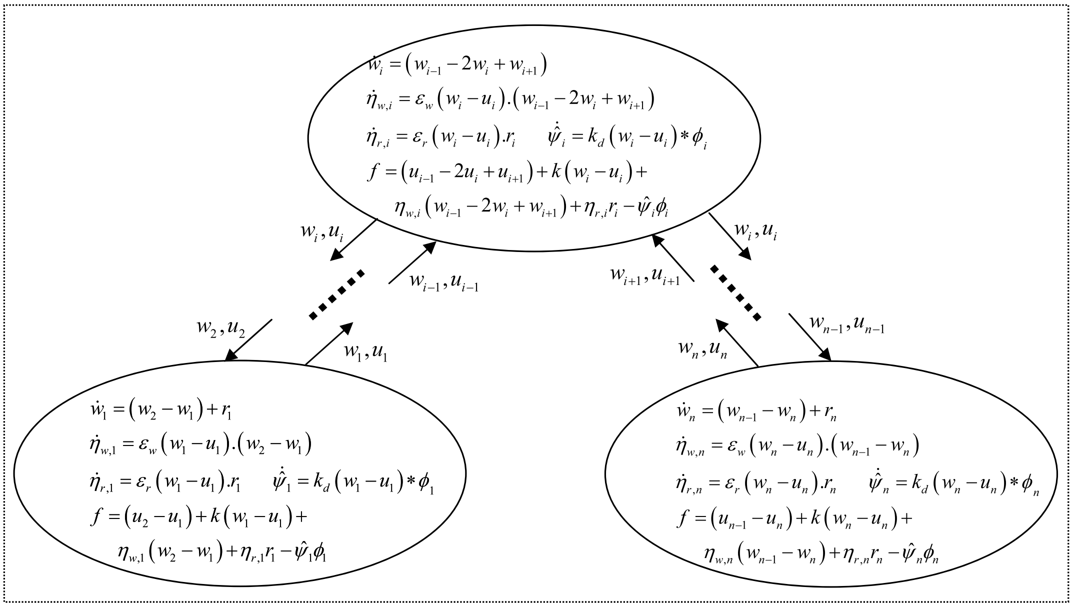

Fig. 2. Localized computation of the desired reference trajectory and the control signal.

Fig. 2 below shows the computation structure for a case of n agents in which agent 1 and agent n are provided the reference inputs. The ith agent communicates only with the $(i-1)$th and the $(i+1)$th agents. Agents $\{2, \ldots, n-1\}$ move solely based on the inputs from their neighbors. As seen, all the computations required for obtaining the control signal are performed locally by each agent.

4.2. Disturbance rejection in the double integrator networks

Similarly to the single integrator case, we now seek to design the control input f in (3), along with the adaptation law, such that $u(x, t)$ of (3) will follow $w(x, t)$ of (6) in spite of uncertainty in $a(x)$.

Proposition 2. *Assume that r, Δw and $\dot{w}$ in the reference model (6) and $\phi(x,t)$ (known time variation of disturbance) in (3) are continuous and bounded for all t. Following definition (8), for the network (3) and the reference model (6) set the control law to be*

$$f = \Delta u - \dot{u} + g - \hat{\psi} \circ \phi \tag{18}$$

with adaptive part

$$g = k_1 e + k_2 \dot{e} + \eta_w \circ \Delta w - \eta_w \circ \dot{w} + \eta_r \circ r, \tag{19}$$

where k_1 and k_2 are positive constants such that $k_2 + 1 > 1/A$, the coefficients η_w and η_r are adapted according to

$$\dot{\eta}_w = \varepsilon_w (e + \dot{e}) \circ (\Delta w - \dot{w}), \qquad \dot{\eta}_r = \varepsilon_r (e + \dot{e}) \circ r, \tag{20}$$

where ε_w and ε_r are positive constants. $\hat{\psi}$ is an observer for ψ updated according to

$$\dot{\hat{\psi}} = k_d (e + \dot{e}) \circ \phi, \tag{21}$$

where k_d is a strictly negative constant. Then, $\|e\|$ tends to zero as $t \to \infty$.

Proof. For the disturbance observer

$$\dot{\tilde{\psi}} = -\dot{\hat{\psi}} = -k_d (e + \dot{e}) \circ \phi, \tag{22}$$

where $\tilde{\psi} = \psi(x) - \hat{\psi}(x)$. Note that

$$\dot{\xi}_w = \dot{\eta}_w = \varepsilon_w (e + \dot{e}) \circ (\Delta w - \dot{w}), \qquad \dot{\xi}_r = \dot{\eta}_r = \varepsilon_r (e + \dot{e}) \circ r. \tag{23}$$

The error $e(x,t)$ evolves according to

$$\begin{aligned}
\ddot{e} &= \ddot{w} - \ddot{u} = \Delta w - \dot{w} + r - \left(a \circ \Delta u - a \circ \dot{u} + a \circ g - a \circ \hat{\psi} \circ \phi + a \circ \psi \circ \phi \right) \\
&= \Delta w - a \circ \Delta w + a \circ \Delta w - \dot{w} + a \circ \dot{w} - a \circ \dot{w} - a \circ \Delta u + a \circ \dot{u} + r \\
&\quad - a \circ (k_1 e + k_2 \dot{e} + \eta_w \circ \Delta w - \eta_w \circ \dot{w} + \eta_r \circ r) - a \circ \tilde{\psi} \circ \phi \\
&= a \circ \left(\frac{[1-a]}{[a]} \right) \circ \Delta w - a \circ \left(\frac{[1-a]}{[a]} \right) \circ \dot{w} - a \circ \eta_w \circ \Delta w + a \circ \eta_w \circ \dot{w} \\
&\quad + a \circ \left(\frac{[1]}{[a]} - \eta_r \right) \circ r + a \circ (\Delta e - \dot{e} - k_1 e - k_2 \dot{e}) - a \circ \tilde{\psi} \circ \phi \\
&= a \circ \left((\eta_w^* - \eta_w) \circ \Delta w - (\eta_w^* - \eta_w) \circ \dot{w} + (\eta_r^* - \eta_r) \circ r \right) + a \circ \left(\Delta e - \dot{e} - k_1 e - k_2 \dot{e} - \tilde{\psi} \circ \phi \right) \\
&= a \circ \left(\Delta e - \dot{e} - k_1 e - k_2 \dot{e} - \xi_w \circ \Delta w + \xi_w \circ \dot{w} - \xi_r \circ r - \tilde{\psi} \circ \phi \right).
\end{aligned} \tag{24}$$

Consider the Lyapunov function

$$\begin{aligned}
V &= \frac{1}{2} \left(\begin{bmatrix} \frac{[1]}{[a]} \circ (\dot{e} + e) \\ -\tilde{\psi}/k_d \end{bmatrix}, \begin{bmatrix} (\dot{e} + e) \\ \tilde{\psi} \end{bmatrix} \right) + \frac{k_1}{2}(e, e) + \frac{1}{2} \left(\left(k_2 - \frac{[1]}{[a]} + 1 \right) e, e \right) \\
&\quad + \frac{1}{2}(-\Delta e, e) + \frac{1}{2\varepsilon_w}(\xi_w, \xi_w) + \frac{1}{2\varepsilon_r}(\xi_r, \xi_r).
\end{aligned}$$

Note that $((k_2 - [1]/[a] + 1)e, e) \geq 0$, since $k_2 + 1 > 1/A$ and $(-\Delta e, e) \geq 0$ according to Lemma 1. It follows from Lemma 2 that

$$\begin{aligned}
\dot{V} &= \left(\frac{[1]}{[a]} \circ (\ddot{e} + \dot{e}), \dot{e} + e \right) - (\dot{\tilde{\psi}}/k_d, \tilde{\psi}) + k_1(\dot{e}, e) \\
&\quad + \left(\left(k_2 - \frac{[1]}{[a]} + 1 \right) \dot{e}, e \right) - (\Delta e, \dot{e}) + \frac{1}{\varepsilon_w}(\dot{\xi}_w, \xi_w) + \frac{1}{\varepsilon_r}(\dot{\xi}_r, \xi_r).
\end{aligned}$$

Further on, it follows from (8), (22), (23) and (24) that

$$\begin{aligned}
\dot{V} &= \left(\Delta e - k_1 e - \left(k_2 - \frac{[1]}{[a]} + 1 \right) \dot{e} - \xi_w \circ \Delta w + \xi_w \circ \dot{w} - \xi_r \circ r - \tilde{\psi} \circ \phi, \dot{e} + e \right) + ((e + \dot{e}) \circ \phi, \tilde{\psi}) \\
&\quad + k_1(\dot{e}, e) + \left(\left(k_2 - \frac{[1]}{[a]} + 1 \right) \dot{e}, e \right) - (\Delta e, \dot{e}) + ((e + \dot{e}) \circ (\Delta w - \dot{w}), \xi_w) + ((e + \dot{e}) \circ r, \xi_r) \\
&= (\Delta e, e) + (\Delta \dot{e}, \dot{e}) - k_1(e, e) - \left(\left(k_2 - \frac{[1]}{[a]} + 1 \right) \dot{e}, \dot{e} \right).
\end{aligned}$$

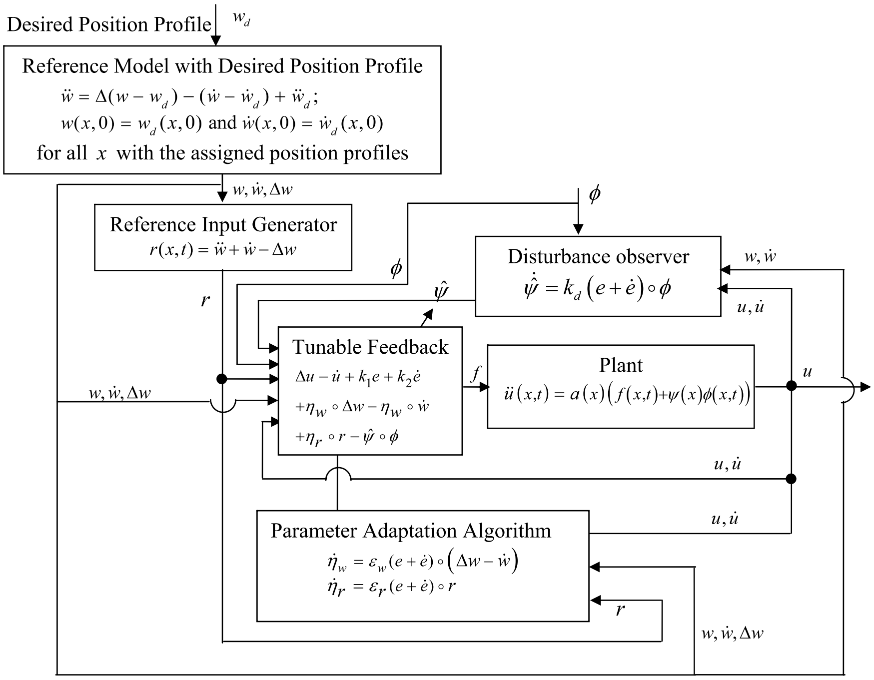

Fig. 3. Implementation schematics of the MRAC law with disturbance rejection for the double integrator network.

Thus $\dot{V} < 0$ unless $(e, e) = 0$. As in the proof of Proposition 1 the error system (22)–(24) is a linear system with bounded time-varying coefficients. Hence all the conditions of Lasalle/Yoshizawa theorem [12, Thm. 8.4, p. 323] are satisfied globally and therefore $\|e\|$ tends to zero as $t \to \infty$. □

With velocity profiles replaced by the position profiles, Remark 1 holds for Proposition 2, as well. Combining the results of this subsection with the double integrator network results of Section 3.2 yields MRAC schematics shown in Fig. 3.

5. Simulations

This section applies the control laws given in Figs. 1 and 3 to networks consisting of three agents. In the reference models considered, only the boundary agents have non-zero reference input, so that the evolution of the middle reference agent is induced by the communication structure. The nature of the latter motion is shown to resemble evolution of parabolic and hyperbolic systems in the single integrator and the double integrator cases, respectively. For simplicity, the variable of interest $z(x, t) \in \mathbb{R}^n$, $x = 1, 2, 3$, corresponding to the xth agent, is denoted by $z_i(t)$, $i \equiv x$. To enhance clarity, notation v and p of the original physical system is retained to denote agent velocity and position, respectively, for both the plant evolution and the reference profiles. To distinguish the latter from one another, subscripts u and r are utilized, respectively. Finally, subscripts x and y are used to indicate the x and y components of the plant and the reference profile variables of interest. An important feature inherited from [5] by the control laws in Figs. 1 and 3 is that their parameter adaptation does not use Δu, and hence is not directly affected by any nonidealities in communication between agents in forming Δu. This makes it less sensitive to communication noise, since no nonideality is involved in forming Δw_r.

5.1. MRAC with disturbance rejection for a single integrator network: Parabolic evolution

In this section, the similarity between the evolution of the reference model and that of a parabolic equation is shown, and the satisfactory performance of the MRAC law for a three-agent single integrator network under disturbance is demonstrated.

Consider a velocity coordination problem in which it is desired that the velocities of three agents moving in a plane and comprising the single integrator network follow the desired velocity profile. Such problems arise, for example, in the context of consensus forming [2] and surveillance. Let the network be governed by

$$\begin{aligned} \dot{v}_{u1} &= f_1 + \psi_{11}\sin(t) + \psi_{12}\cos(t), \\ \dot{v}_{u2} &= 2f_2 + \psi_2\sin(2t), \\ \dot{v}_{u3} &= 4f_3 + \psi_3\cos(t), \end{aligned} \quad (25)$$

where $v_{ui} = [v_{uix}\, v_{uiy}]^T \in \mathbb{R}^2$ is the two-component velocity of the ith agent, and the network information structure be given by the direct communication between agents 1 and 2 and between agents 2 and 3. The constant parameter in the disturbance terms are

$$\psi_{11} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad \psi_{12} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad \psi_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \qquad \psi_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}. \tag{26}$$

Hence the reference model (4) takes the form

$$\begin{aligned} \dot{v}_{r1} &= v_{r2} - v_{r1} + r_1, \\ \dot{v}_{r2} &= v_{r3} - 2v_{r2} + v_{r1} + r_2, \\ \dot{v}_{r3} &= v_{r2} - v_{r3} + r_3. \end{aligned} \tag{27}$$

Let the desired velocity profile be given by

$$v_{d1} = \begin{bmatrix} \begin{Bmatrix} 0, & 0 \le t < 10 \\ t-10, & 10 \le t < 11 \\ 1, & t \ge 11 \end{Bmatrix} \\ 0 \end{bmatrix}, \qquad v_{d3} = \begin{bmatrix} \begin{Bmatrix} 0, & 0 \le t < 10 \\ t-10, & 10 \le t < 11 \\ 1, & t \ge 11 \end{Bmatrix} \\ 1 \end{bmatrix}, \tag{28}$$

where $v_{di} = [v_{dix}(t)\, v_{diy}(t)]^T \in \mathbb{R}^2$, $i = 1, 3$, represents the desired velocity for the ith agent. The corresponding reference input computed using (5) and the desired profile is given by

$$\begin{aligned} r_1 &= \begin{bmatrix} \dot{v}_{d1x} + v_{d1x} - v_{r2x} \\ \dot{v}_{d1y} + v_{d1y} - v_{r2y} \end{bmatrix}, \qquad r_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ r_3 &= \begin{bmatrix} \dot{v}_{d3x} + v_{d3x} - v_{r2x} \\ \dot{v}_{d3y} + v_{d3y} - v_{r2y} \end{bmatrix}. \end{aligned} \tag{29}$$

Substituting these r_1 and r_3 into (27) yields the following reference model with decentralized velocity profile

$$\begin{aligned} \dot{v}_{r1} &= -v_{r1} + \dot{v}_{d1} + v_{d1}, \\ \dot{v}_{r2} &= v_{r3} - 2v_{r2} + v_{r1}, \\ \dot{v}_{r3} &= -v_{r3} + \dot{v}_{d3} + v_{d3}, \end{aligned} \tag{30}$$

where for the second agent no desired velocity profile is specified and the reference input is 0. The velocity of this agent is determined purely by the choice of communication structure used in Eq. (27). Setting $v_{ri}(t = 0) = v_{di}(t = 0)$, $i = 1, 3$, ensures that $v_{ri}(t) = v_{di}(t)$, $i = 1, 3$, i.e. the exact following of the assigned velocity profile by agents 1 and 3. Fig. 4(a) depicts the resulting reference velocity profile. The reference inputs r_1 and r_3 in (29) used in the control law (9)–(11) are calculated using v_{r2} determined from solving (30).

There are 8 unknown observer variables associated with the disturbance in (25). The parameter update law (12) for this problem takes the form

$$\begin{bmatrix} \dot{\hat{\psi}}_{11} \\ \dot{\hat{\psi}}_{12} \\ \dot{\hat{\psi}}_2 \\ \dot{\hat{\psi}}_3 \end{bmatrix} = \begin{bmatrix} -10(v_{r1} - v_{u1})\sin(t) \\ -10(v_{r1} - v_{u1})\cos(t) \\ -10(v_{r2} - v_{u2})\sin(2t) \\ -10(v_{r3} - v_{u3})\cos(t) \end{bmatrix}. \tag{31}$$

The adaptive control law defined in Proposition 1 takes the form

$$\begin{aligned} f_1 &= (v_{u2} - v_{u1}) + k(v_{d1} - v_{u1}) + \eta_{w,1} \circ (v_{r2} - v_{d1}) + \eta_{r,1} \circ r_1 - \hat{\psi}_{11}\sin(t) - \hat{\psi}_{12}\cos(t), \\ f_2 &= (v_{u3} - v_{u2}) + (v_{u1} - v_{u2}) + k(v_{r2} - v_{u2}) + \eta_{w,2} \circ ((v_{d3} - v_{r2}) + (v_{d1} - v_{r2})) - \hat{\psi}_2 \sin(2t), \\ f_3 &= (v_{u2} - v_{u3}) + k(v_{d3} - v_{u3}) + \eta_{w,3} \circ (v_{r2} - v_{d3}) + \eta_{r,3} \circ r_3 - \hat{\psi}_3 \cos(t), \end{aligned} \tag{32}$$

with

$$\begin{aligned} \dot{\eta}_{w,1} &= \varepsilon_w (v_{d1} - v_{u1}) \circ (v_{r2} - v_{d1}), \\ \dot{\eta}_{w,2} &= \varepsilon_w (v_{r2} - v_{u2}) \circ ((v_{d3} - v_{r2}) + (v_{d1} - v_{r2})), \\ \dot{\eta}_{w,3} &= \varepsilon_w (v_{d3} - v_{u3}) \circ (v_{r2} - v_{d3}), \\ \dot{\eta}_{r,1} &= \varepsilon_r (v_{d1} - v_{u1}) \circ r_1, \qquad \dot{\eta}_{r,3} = \varepsilon_r (v_{d3} - v_{u3}) \circ r_3. \end{aligned} \tag{33}$$

In both (32) and (33) terms r_2 and $\eta_{r,2} r_2$ are omitted.

Fig. 4(b) depicts the velocity profiles of the three agents comprising the controlled network in the v_u plane, with initial conditions

$$v_{u1}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad v_{u2}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad v_{u3}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

subject to the control law (32) and (33) with $\eta_{w,i}(0) = \eta_{r,i}(0) = 0$ for $i = 1, 2, 3$ and $\hat{\psi}_{11}(0) = \hat{\psi}_{12}(0) = \hat{\psi}_2(0) = \hat{\psi}_3(0) = 0$. The adaptively controlled single integrator network is seen to track the reference velocity profile asymptotically.

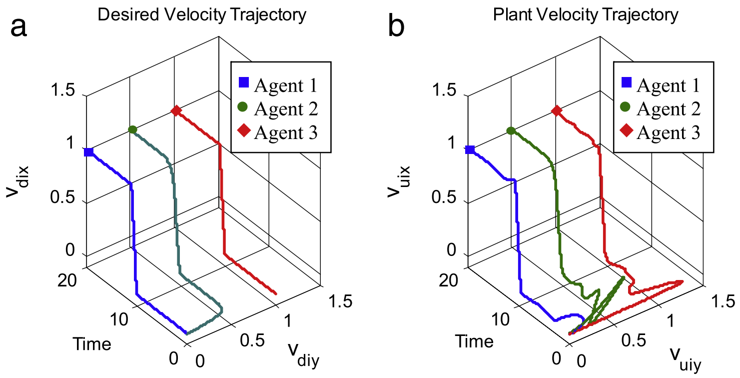

Fig. 4. The reference velocity profile (graph (a)) and the corresponding single integrator plant velocity profile (graph (b)) attained under disturbance by the adaptive control law.

5.2. MRAC with disturbance rejection for a double integrator network: Hyperbolic evolution

In this section, the similarity between the evolution of the reference model and that of a hyperbolic equation is shown, and the satisfactory performance of the MRAC law for a three-agent double integrator network with decentralized position profile assignment is demonstrated under disturbance. Consider a three-agent network moving in a plane with the task of tracking the reference position profile. Let the network be governed by

$$\begin{aligned}\ddot{p}_{u1} &= f_1 + \psi_{11}\sin(4\sin(t)) + \psi_{12}\cos(4\sin(t)),\\ \ddot{p}_{u2} &= 2f_2 + \psi_2\sin(t),\\ \ddot{p}_{u3} &= 4f_3 + \psi_3\sin(2t),\end{aligned} \tag{34}$$

where $p_{ui} = [p_{uix}(t)\, p_{uiy}(t)] \in \mathbb{R}^2$ is the two-component position of the ith agent, and have the same topology as that of the system considered in Section 5.1. The constant parameter in the disturbance terms are

$$\psi_{11} = \begin{bmatrix}1\\1\end{bmatrix},\qquad \psi_{12} = \begin{bmatrix}1\\2\end{bmatrix},\qquad \psi_2 = \begin{bmatrix}2\\4\end{bmatrix},\qquad \psi_3 = \begin{bmatrix}2\\4\end{bmatrix}. \tag{35}$$

The reference model (6) in this case takes the form

$$\begin{aligned}\ddot{p}_{r1} &= p_{r2} - p_{r1} - \dot{p}_{r1} + r_1,\\ \ddot{p}_{r2} &= p_{r3} - 2p_{r2} + p_{r1} - \dot{p}_{r2} + r_2,\\ \ddot{p}_{r3} &= p_{r2} - p_{r3} - \dot{p}_{r3} + r_3.\end{aligned} \tag{36}$$

Let the desired position profile be given by

$$\begin{bmatrix}p_{d1x}\\p_{d1y}\end{bmatrix} = \begin{bmatrix}\{-10+t,\ t\geq 0\}\\ \begin{Bmatrix}1, & 0\leq t<8\\ 6(t-8)^5 - 15(t-8)^4 + 10(t-8)^3 + 1, & 8\leq t<9\\ 2, & t\geq 9\end{Bmatrix}\end{bmatrix},$$

$$\begin{bmatrix}p_{d3x}\\p_{d3y}\end{bmatrix} = \begin{bmatrix}\{-10+t,\ t\geq 0\}\\ \begin{Bmatrix}-1, & 0\leq t<8\\ 6(t-8)^5 - 15(t-8)^4 + 10(t-8)^3 + 1, & 8\leq t<9\\ 0, & t\geq 9\end{Bmatrix}\end{bmatrix},$$

where $p_{di} = [p_{dix}(t)\, p_{diy}(t)] \in \mathbb{R}^2$ represents the two-component desired position of the ith agent. The corresponding reference input computed using (7) and the desired profile is given by

$$\begin{aligned} r_1 &= \begin{bmatrix}\ddot{p}_{d1x} + \dot{p}_{d1x} + p_{d1x} - p_{r2x}\\ \ddot{p}_{d1y} + \dot{p}_{d1y} + p_{d1y} - p_{r2y}\end{bmatrix},\qquad r_2 = \begin{bmatrix}0\\0\end{bmatrix},\\ r_3 &= \begin{bmatrix}\ddot{p}_{d3x} + \dot{p}_{d3x} + p_{d3x} - p_{r2x}\\ \ddot{p}_{d3y} + \dot{p}_{d3y} + p_{d3y} - p_{r2y}\end{bmatrix}.\end{aligned} \tag{37}$$

Substituting these r_1 and r_3 into (36) yields reference model with decentralized position profile assignment given by

$$\begin{aligned}\ddot{p}_{r1} &= -p_{r1} - \dot{p}_{r1} + \ddot{p}_{d1} + \dot{p}_{d1} + p_{d1},\\ \ddot{p}_{r2} &= p_{r3} - 2p_{r2} + p_{r1} - \dot{p}_{r2},\\ \ddot{p}_{r3} &= -p_{r3} - \dot{p}_{r3} + \ddot{p}_{d3} + \dot{p}_{d3} + p_{d3}.\end{aligned} \tag{38}$$

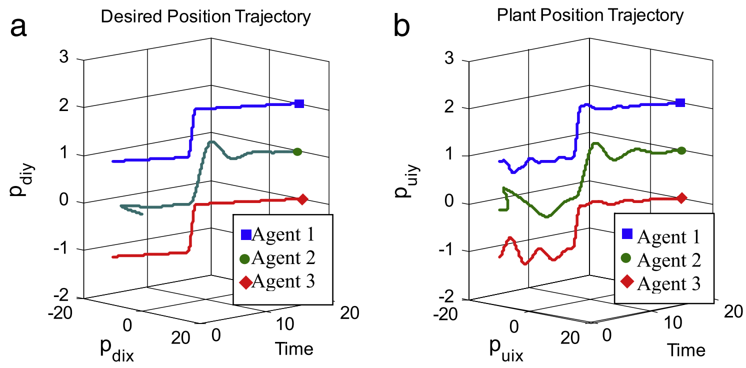

Fig. 5. The reference position profile (graph (a)) and the corresponding position profile of the double integrator plant (graph (b)) attained under disturbance by the adaptive control law.

Setting $p_{ri}(t=0) = p_{di}(t=0), \dot{p}_{ri}(t=0) = \dot{p}_{di}(t=0), i = 1,3$, ensures that $p_{ri}(t) = p_{di}(t),\ i = 1,3$. Note that for the second agent no desired position profile is specified and the reference input is 0. Its position is determined solely by the choice of communication structure used in Eq. (36). Fig. 5(a) depicts the resulting network reference position profile. The reference inputs r_1 and r_3 in (37) used in the control law (18)–(20) are calculated using p_{r2} determined from solving (38). There are 8 unknown observer variables associated with the disturbance in (34). The parameter update law (21) for this problem takes the form

$$\begin{bmatrix} \dot{\hat{\psi}}_{11} \\ \dot{\hat{\psi}}_{12} \\ \dot{\hat{\psi}}_{2} \\ \dot{\hat{\psi}}_{3} \end{bmatrix} = \begin{bmatrix} -2(\dot{v}_{r1} - \dot{v}_{u1} + v_{r1} - v_{u1})\sin(4\sin(t)) \\ -2(\dot{v}_{r1} - \dot{v}_{u1} + v_{r1} - v_{u1})\cos(4\sin(t)) \\ -2(\dot{v}_{r2} - \dot{v}_{u2} + v_{r2} - v_{u2})\sin(t) \\ -2(\dot{v}_{r3} - \dot{v}_{u3} + v_{r3} - v_{u3})\sin(2t) \end{bmatrix}. \tag{39}$$

Omitting the terms involving r_2 as a factor, the adaptive control law defined in Proposition 2 takes the form

$$\begin{aligned} f_1 &= (p_{u2} - p_{u1}) - \dot{p}_{u1} + k_1(p_{d1} - p_{u1}) + k_2(\dot{p}_{d1} - \dot{p}_{u1}) + \eta_{w,1} \circ (p_{r2} - p_{d1}) - \eta_{w,1} \circ \dot{p}_{d1} \\ &\quad + \eta_{r,1} \circ r_1 - \hat{\psi}_{11}\sin(4\sin(t)) - \hat{\psi}_{12}\cos(4\sin(t)), \\ f_2 &= (p_{u3} - p_{u2}) + (p_{u1} - p_{u2}) - \dot{p}_{u2} + k_1(p_{r2} - p_{u2}) + k_2(\dot{p}_{r2} - \dot{p}_{u2}) \\ &\quad + \eta_{w,2} \circ ((p_{d3} - p_{r2}) + (p_{d1} - p_{r2})) - \eta_{w,2} \circ \dot{p}_{r2} - \hat{\psi}_2 \sin(t), \\ f_3 &= (p_{u2} - p_{u3}) - \dot{p}_{u3} + k_1(p_{d3} - p_{u3}) + k_2(\dot{p}_{d3} - \dot{p}_{u3}) \\ &\quad + \eta_{w,3} \circ (p_{r2} - p_{d3}) - \eta_{w,3} \circ \dot{p}_{d3} + \eta_{r,3} \circ r_3 - \hat{\psi}_3 \sin(2t), \end{aligned} \tag{40}$$

with

$$\begin{aligned} &\dot{\eta}_{w,1} = \varepsilon_w((p_{d1} - p_{u1}) + (\dot{p}_{d1} - \dot{p}_{u1})) \circ (p_{r2} - p_{d1} - \dot{p}_{d1}), \\ &\dot{\eta}_{w,2} = \varepsilon_w((p_{r2} - p_{u2}) + (\dot{p}_{r2} - \dot{p}_{u2})) \circ ((p_{d3} - p_{r2}) + (p_{d1} - p_{r2}) - \dot{p}_{r2}), \\ &\dot{\eta}_{w,3} = \varepsilon_w((p_{d3} - p_{u3}) + (\dot{p}_{d3} - \dot{p}_{u3})) \circ (p_{r2} - p_{d3} - \dot{p}_{d3}), \\ &\dot{\eta}_{r,1} = \varepsilon_r((p_{d1} - p_{u1}) + (\dot{p}_{d1} - \dot{p}_{u1})) \circ r_1, \\ &\dot{\eta}_{r,3} = \varepsilon_r((p_{d3} - p_{u3}) + (\dot{p}_{d3} - \dot{p}_{u3})) \circ r_3. \end{aligned} \tag{41}$$

Fig. 5(b) depicts the position profiles of the three agents comprising the controlled network in the p_u plane, with initial conditions

$$p_{u1}(0) = \begin{bmatrix} -10 \\ 1 \end{bmatrix}, \qquad p_{u2}(0) = \begin{bmatrix} -10 \\ 0 \end{bmatrix}, \qquad p_{u3}(0) = \begin{bmatrix} -10 \\ -1 \end{bmatrix},$$

$$\dot{p}_{ui}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad i = 1, 2, 3,$$

subject to the control (40), (41) with zero initial values for all control parameters. It is seen that the algorithm developed provides successful tracking under agent parameter uncertainty.

6. Conclusion

A well-posed globally stable PDE-based model reference adaptive control setting of [5] is extended to control of uncertain heterogeneous multiagent networks, evolving under the influence of external disturbance, described by partial difference equations on graphs. The reference models with the decentralized motion assignment are introduced. The algorithms developed are augmented with the capability to track the desired profiles of the variables of interest. The setting proposed paves the way for considering control of uncertain multiagent networks under disturbances and switching topologies.

Acknowledgements

The present work was supported by the NSF Grants CMS-0324630, ECS-0501407, and CMS-0449319. Shu Zhang is gratefully acknowledged for his help with the figures and participation in filling in the details of the proof of Proposition 1.

This work is based on the invited lecture presented at the 2008 World Congress of Nonlinear Analysts, Orlando, Florida, July 2–9, 2008.

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