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Hamiltonian Mechanics and Nonlinear Dynamics of a Body Subject to Time-Varying Gyroscopic and Potential Forces

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Abstract— **We consider the planar dynamics of a body under the simultaneous influence of a force given by the spatial gradient of a potential function and a force that remains perpendicular to the body's translational velocity. Either force can vary with time. We describe a physical example of such a system and analyze the structure of the underlying equations of motion, demonstrating that they exemplify a special class of almost Hamiltonian systems. We show that when the potential function varies periodically with time, the momentum of the body can evolve chaotically.**

I. INTRODUCTION

The dynamic behavior of a body under the influence of a force derived from a spatial potential is a canonical problem in mechanics. Gravitational forces, electromagnetic forces, and spring forces may all be realized in this way; such forces may vary explicitly with time as well as position.

A *gyroscopic force* on a moving body is a force that acts perpendicular to the body's velocity. Like forces derived from spatial potentials, gyroscopic forces arise in a variety of contexts, and include the most elemental forms of aerodynamic and hydrodynamic lift. A gyroscopic force does no work, but may still influence a body's trajectory. In the classic optimal control problem concerning the motion of a particle moving in the plane at constant speed along a path with time-varying radius of curvature [1], the control input may be regarded as a gyroscopic force on a moving mass, inducing the mass to veer more or less tightly to the left or right.

The present paper addresses the mechanics and dynamics of systems in which spatial potentials and gyroscopic forces interact. We focus on a simple toy problem that may be realized physically in the manner described below, but we consider this problem to typify a broad class of problems of mathematical and engineering interest.

Imagine a circular body moving through a planar fluid. The body is a hollow cylinder, coupled to an internal rotor such that the surface of the cylinder can be made to spin by counter-spinning the rotor. As the body translates, it will experience a gyroscopic lift force that depends on its spinning speed, each perhaps varying over time. Now subject this body to an external potential. The potential may be that of gravity, acting on a cylindrical depth charge that spins to navigate as it sinks, or the potential may represent exogenous control. The resulting system is depicted in Fig. 1.

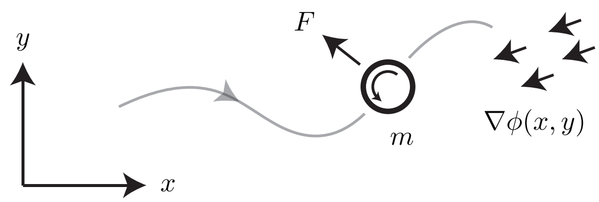

Fig. 1. A body with mass m moving in the (x, y) plane under the influence of a gyroscopic force with magnitude F and a force equal to the spatial gradient of a function $\phi(x, y)$. When F is positive, the gyroscopic force is directed 90° counterclockwise from the body's velocity. Either force may vary explicitly with time.

The system in Fig. 1 exhibits several interesting features. In the absence of restrictions on F and ϕ, first of all, the equations of motion for the body are endowed with an *almost Hamiltonian* structure, in a sense to be defined in Section II. Under certain assumptions regarding F and ϕ — specifically, assumptions consistent with the idea that F represents a lift force — system trajectories are in one-to-one correspondence with trajectories of a reduced-order system that is properly Hamiltonian.

In Section III, we discuss the geometric mechanics underlying this reduction of dynamics. We show that the reduced system preserves a notion of volume in the space of momenta and is therefore an example of a *Liouville system* [2]. For systems with two degrees of freedom, the notions of Liouville and Hamiltonian dynamical systems coincide, whereas in higher dimensions the class of Liouville systems is more general. We derive a geometric criterion for an almost Hamiltonian system to be reducible to Liouville form, and we discuss two examples: the gyroscopic system of Fig. 1 and the dynamics of a charged particle in a time-dependent but spatially constant electromagnetic field.

When F or ϕ depends periodically on time, the linear momentum of the body in Fig. 1 may evolve chaotically. We demonstrate this in Section IV by showing that stroboscopic Poincaré sections of the momentum space exhibit fractal structure.

II. EQUATIONS OF MOTION

Applied to the body in Fig. 1, Newton's second law indicates that

$$\begin{bmatrix}\dot{p}_x \\ \dot{p}_y\end{bmatrix} = \frac{F}{\sqrt{p_x^2 + p_y^2}}\begin{bmatrix}-p_y \\ p_x\end{bmatrix} + \begin{bmatrix}\partial\phi/\partial x \\ \partial\phi/\partial y\end{bmatrix}, \tag{1}$$

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where

$$p_x = m\dot{x}, \qquad p_y = m\dot{y} \tag{2}$$

are the two components of the body's linear momentum.

A. Almost Hamiltonian Structure

We may regard x and y as coordinates on the configuration manifold $Q = \mathbb{R}^2$ for the body in Fig. 1, and regard p_x and p_y as corresponding coordinates induced on each cotangent space T_q^*Q. The evolution of the system described by (1) and (2) may be equated with the flow along the vector field

$$X_H = \frac{p_x}{m}\frac{\partial}{\partial x} + \frac{p_y}{m}\frac{\partial}{\partial y} + \left(-\frac{F}{\sqrt{p_x^2 + p_y^2}} p_y + \frac{\partial \phi}{\partial x} \right) \frac{\partial}{\partial p_x} + \left(\frac{F}{\sqrt{p_x^2 + p_y^2}} p_x + \frac{\partial \phi}{\partial y} \right) \frac{\partial}{\partial p_y}$$

on T^*Q.

In terms of the scalar function

$$H : T^*Q \to \mathbb{R} : (x, y, p_x, p_y) \mapsto \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \phi(x, y)$$

and the two-form

$$\Omega = dx \wedge dp_x + dy \wedge dp_y + \frac{Fm}{\sqrt{p_x^2 + p_y^2}} dx \wedge dy, \tag{3}$$

this vector field satisfies

$$\Omega(X_H, w) = \langle dH, w \rangle \tag{4}$$

for every $w \in T_{(q,p)}T^*Q$. The quantity H is clearly the total kinetic and potential energy of the body. If Ω were a symplectic form on T^*Q, then (4) would establish X_H as the Hamiltonian vector field associated with H, but Ω fails the requirement that a symplectic form be closed unless F is a (possibly time-varying) scalar multiple of $\sqrt{p_x^2 + p_y^2}$.

Equivalently, (1) and (2) may be written in the form

$$\dot{x} = \{x, H\}_\Omega, \qquad \dot{y} = \{y, H\}_\Omega$$
$$\dot{p}_x = \{p_x, H\}_\Omega, \qquad \dot{p}_y = \{p_y, H\}_\Omega$$

in terms of the binary operation

$$\{f, g\}_\Omega = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p_x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial p_y} + \frac{\partial f}{\partial p_x}\left(-\frac{\partial g}{\partial x} - \frac{Fm}{\sqrt{p_x^2 + p_y^2}}\frac{\partial g}{\partial p_y} \right) + \frac{\partial f}{\partial p_y}\left(-\frac{\partial g}{\partial y} + \frac{Fm}{\sqrt{p_x^2 + p_y^2}}\frac{\partial g}{\partial p_x} \right)$$

on $C^\infty(T^*Q)$. This operation fails to satisfy the Jacobi identity required of a Poisson bracket unless F is a scalar multiple of $\sqrt{p_x^2 + p_y^2}$.

B. Gyroscopic Lift Force

It was determined experimentally in [3] that within a certain range of Reynolds numbers, the magnitude of the lift on a spinning cylinder translating steadily through a planar fluid depends quadratically on the cylinder's translational speed and linearly on its spinning speed. It was shown in [4] that within an overlapping range of Reynolds numbers — corresponding, in particular, to the navigation of a small robotic vehicle in a laboratory pool — the same dependencies obtain at least coarsely in unsteady flow.

With the substitution

$$F = \lambda\left(p_x^2 + p_y^2\right),$$

the equations of motion (1) become

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = \lambda\sqrt{p_x^2 + p_y^2}\begin{bmatrix} -p_y \\ p_x \end{bmatrix} + \begin{bmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \end{bmatrix}. \tag{5}$$

Here λ may be interpreted as the time-varying coefficient of lift of the body in Fig. 1, proportional to its spinning speed. It's intuitively clear — and easily verified by inspection — that if $\phi = 0$ identically and λ is held constant, then the magnitude $\sqrt{p_x^2 + p_y^2}$ of the body's forward momentum will be conserved, since the force it experiences is always perpendicular to its velocity. With these assumptions, the equations of motion above were obtained via Hamiltonian reduction in [5], framed not as the equations of motion for a steadily spinning cylinder but as those for a cylinder supporting a constant fluid *circulation* around its boundary. It's a basic result in inviscid hydrodynamics that the lift on a translating body is proportional in magnitude to both the body's translational speed and the circulation around its boundary [6].

C. Reduced Hamiltonian Structure

If the spatial potential in (5) takes the form $\phi = Ax + By$, where A and B are free to depend on time but not on x, y, p_x, or p_y, then the evolution of the system described by (5) may be equated with the flow along the vector field

$$X_h = \left(-\lambda\sqrt{p_x^2 + p_y^2}\, p_y + A \right)\frac{\partial}{\partial p_x} + \left(\lambda\sqrt{p_x^2 + p_y^2}\, p_x + B \right)\frac{\partial}{\partial p_y}$$

in the plane $\mathbb{R}^2$ of linear momenta. The potential function associated with gravitational potential energy near the surface of the Earth fits this form, for instance, further simplified by its time invariance.

In terms of the scalar function

$$h : \mathbb{R}^2 \to \mathbb{R} : (p_x, p_y) \mapsto -\frac{1}{3}\lambda\left(p_x^2 + p_y^2\right)^{3/2} + Ap_y - Bp_x$$

and the canonical symplectic form $\omega = dp_x \wedge dp_y$ on $\mathbb{R}^2$, the vector field X_h satisfies

$$\omega(X_h, w) = \langle dh, w \rangle$$

for arbitrary $w \in T_{(p_x,p_y)}\mathbb{R}^2$. Equivalently, the equations governing the evolution of the body's linear momenta may be written in the form

$$\dot{p}_x = \{p_x, h\}_\omega, \qquad \dot{p}_y = \{p_y, h\}_\omega$$

relative to the canonical Poisson bracket

$$\{f, g\}_\omega = \frac{\partial f}{\partial p_x}\frac{\partial g}{\partial p_y} - \frac{\partial f}{\partial p_y}\frac{\partial g}{\partial p_x}$$

on $C^\infty(\mathbb{R}^2)$. The equations governing the evolution of the body's linear momentum are thus properly Hamiltonian when $\phi = Ax + By$.

III. LIOUVILLE SYSTEMS

We now place the observations of Section II in a broader context. We start from the observation that the system detailed in Section II fits the general form of systems that are called *Liouville systems*. The defining feature of such systems is that they preserve a *volume form* in the space of momenta, so that they satisfy a form of Liouville's theorem on the preservation of phase volume. Liouville systems were introduced in [2], but to the best of our knowledge the application to almost Hamiltonian systems, and to dynamical systems of the form discussed in Section II in particular, is new.

For systems with two degrees of freedom, the concepts of volume and symplectic forms coincide, and consequently we will recover the reduced symplectic description of Section II-C. For systems with more than two degrees of freedom, the two concepts are different and there is in general no reduced symplectic description.

Given that both the gyroscopic force and the linear potential of a Liouville system are allowed to depend explicitly on time, it is natural to consider *controlled* Liouville systems. A geometric theory of controllability for symplectic control systems (i.e., two-dimensional Liouville systems) has been developed in [7], while the extension to higher-dimensional Liouville systems will be the subject of future work.

A. Outline

Roughly speaking, in two dimensions our method proceeds as follows: given the almost Hamiltonian structure defined by H and Ω, we observe that, in the case of a linear potential $\phi = Ax + By$, the momentum equations (1) determining p_x and p_y decouple from the reconstruction equations (2). The momentum equations can be given a Hamiltonian structure with respect to the symplectic form $\omega = dp_x \wedge dp_y$ if the right-hand side of (1) is curl-free:

$$\frac{\partial}{\partial p_y}\left(\frac{F}{\sqrt{p_x^2 + p_y^2}}(-p_y)\right) - \frac{\partial}{\partial p_x}\left(\frac{F}{\sqrt{p_x^2 + p_y^2}}p_x\right) = 0,$$

and this is equivalent to the requirement that F depend on p_x and p_y through the radial distance $p_x^2 + p_y^2$ only. This is the case for the special gyroscopic force considered in Section II-B. We now generalize these observations to the case of dynamical systems with more than two degrees of freedom.

B. Hamiltonian Equations

For the remainder of this section, we let the configuration space Q be $\mathbb{R}^n$. We denote the points in Q by $q := (q^1, \ldots, q^n)^T$ and we identify T^*Q with $Q \times P$, where $P \cong \mathbb{R}^n$ is referred to as the *momentum space*, whose elements are denoted by $p := (p_1, \ldots, p_n)^T$. We employ the Einstein summation convention throughout: whenever an index appears twice, once as a subscript and once as a superscript, an implicit summation over that index is understood.

We assume that we are given a Hamiltonian H of the form

$$H(q,p) = \frac{1}{2}p^T M^{-1} p + A^T q,$$

where the mass matrix M is constant and invertible, and the column vector $A \in \mathbb{R}^n$ may depend on time. Note that M determines a metric tensor on Q, given by $M = M_{ij}dq^i \otimes dq^j$, and that A can be viewed as a one-form on Q, given by $A = A_i dq^i$.

We assume furthermore that the system is acted upon by a gyroscopic force, which we model as a differential two-form F whose coefficient functions $F_{ij}(p)$ depend on p but not on q:

$$F = \frac{1}{2}F_{ij}(p)dq^i \wedge dq^j. \tag{6}$$

The coefficient matrix is antisymmetric: $F_{ij} = -F_{ji}$. Note that in comparison with (3) we have absorbed the factor $m\sqrt{p_x^2 + p_y^2}$ into the definition of F. We then define

$$\Omega := dq^i \wedge dp_i + \frac{1}{2}F_{ij}(p)dq^i \wedge dq^j.$$

Note that this form is again not closed, unless $F_{ij}(p)$ is constant. With respect to this almost symplectic form, Hamilton's equations become

$$\dot{q} = M^{-1}p \tag{7}$$

and

$$\dot{p} = -A + F(p)M^{-1}p. \tag{8}$$

Observe in particular that the latter equation is independent of the position coordinates q.

C. Aspects of Riemannian Geometry

Before proceeding with the analysis of (8), we recall some foundational aspects of Riemannian geometry (see [8] for more information). The inverse mass matrix M^{-1} determines a (constant) Riemannian metric on P, given by $M^{-1} = (M^{-1})^{ij}dp_i \otimes dp_j$, and we denote the induced Riemannian volume form by ω:

$$\omega := (\det M)^{-1/2} dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n.$$

We also introduce the $(n-1)$-forms $\omega^{(i)}$, given by

$$\begin{aligned}\omega^{(i)} &= \mathbf{i}_{\partial/\partial p_i}\omega \\ &= (-1)^{i+1}(\det M)^{-1/2} dp_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_n.\end{aligned}$$

In other words, up to sign $\omega^{(i)}$ is ω with the dp_i-term removed. The Hodge star relative to the mass matrix is denoted by $* : \omega^k(P) \to \omega^{n-k}(P)$ and is determined in coordinates by

$$*(dp_k) = M_{kl}(\det M)\omega^{(l)}.$$

We also define the map $J : T(T^*Q) \to T(T^*Q)$, given in coordinates by

$$J = M_{ij} dq^i \otimes \frac{\partial}{\partial p_j} - (M^{-1})^{ij} dp_i \otimes \frac{\partial}{\partial q^j}.$$

It can easily be checked that $J^2 = -1$, making J into an *almost complex structure* on T^*Q. The map J can be given intrinsic meaning in terms of the symplectic structure and the metric induced by M (see [8]). Here, we use J and its dual J^* to transform (co-)vectors on Q into (co-)vectors on P by noting that

$$J\left(\frac{\partial}{\partial q^j}\right) = M_{ij}\frac{\partial}{\partial p_j} \quad \text{and} \quad J^*(dq^i) = -(M^{-1})^{ij} dp_j.$$

Using the map J, we may transform the gyroscopic two-form F on T^*Q, given in (6), into a two-form G on P:

$$G(v, w) = F(J(v), J(w)),$$

for all $v, w \in TP$. In coordinates, G is given by $G^{ij} dp_i \wedge dp_j$, where

$$G^{ij}(p) = (M^{-1})^{ik} F_{kl}(p)(M^{-1})^{lj}.$$

We also define the one-form B on P, defined in the same way by $B(v) = A(J(v)) = (J^*A)(v)$ for all $v \in TP$. In coordinates, we have $B = B^i dp_i$, with

$$B^i = -(M^{-1})^{ij} A_j.$$

D. Liouville Systems

We now return to the analysis of the momentum equation (8). By itself, this equation is in general not Hamiltonian: one way to see this is to observe, for instance, that the momenta p are elements of $\mathbb{R}^n$, which, for n odd, does not have a symplectic form. In this section, we will show that instead the p-equations preserve the volume form ω in $\mathbb{R}^n$.

The momentum equation can be written as

$$\dot{p}_i \omega^{(i)} = (B^j + G^{jk} p_k) M_{ij} \omega^{(i)}.$$

The right-hand side can be written using the Hodge star as $*(B + \mathbf{i}_p G)$, while the left hand side can be written as

$$\mathbf{i}_{X_P}\omega, \quad \text{where} \quad X_p = \dot{p}_i \frac{\partial}{\partial p_i}.$$

The momentum equation then becomes

$$\mathbf{i}_{X_P}\omega = *(B + \mathbf{i}_p G).$$

So far, we have merely rewritten the momentum equation. We now make the additional assumption that the exterior derivative of the right-hand side vanishes:

$$\mathbf{d} * (B + \mathbf{i}_p G) = 0. \tag{9}$$

As $\mathbb{R}^n$ is topologically trivial, this implies that there exists an $(n-2)$-form $\mathcal{K}$ such that

$$*(B + \mathbf{i}_p G) = \mathbf{d}\mathcal{K}. \tag{10}$$

We will refer to $\mathcal{K}$ as the *Hamiltonian form*. The equation (8) then becomes

$$\mathbf{i}_{X_P}\omega = \mathbf{d}\mathcal{K}. \tag{11}$$

While this expression is similar to the symplectic formulation of Hamilton's equations, the difference is that ω is a volume form instead of a symplectic two-form, while $\mathcal{K}$ is no longer a function, but a form of degree $n-2$. The equation (11), with ω a volume form, presents an example of a Liouville system. In general, equations of this form, in which the symplectic form is replaced by a form of higher degree, have been studied in the literature under the name of *multisymplectic equations* [9].

Finally, the condition (9) can be written in a more manageable form. Using the definition of the *codifferential* $\delta = *\mathbf{d}*$, and the fact that B is constant, we have that (9) is equivalent to

$$\delta(\mathbf{i}_p G) = 0. \tag{12}$$

A small calculation, using the fact that $dp_i \wedge \omega^{(j)} = \delta_i^j (\det M)^{-1/2}\omega$, shows that this is equivalent with the scalar condition

$$M_{ij}\frac{\partial G^{jk}}{\partial p_i} p_k = 0, \tag{13}$$

which in turn can be written as $\mathbf{i}_p(\delta G) = 0$.

E. Properties of Liouville Systems

It is not hard to show that the flow of the vector field X_P satisfying (11) is volume-preserving:

$$\pounds_{X_P}\omega = \mathbf{d}\mathbf{i}_{X_P}\omega = \mathbf{d}^2\mathcal{K} = 0,$$

where $\pounds_{X_P}$ is the Lie derivative along X_P. The flow of X_P is therefore volume preserving in P and satisfies a Liouville-type theorem in P. As a result, techniques from the theory of measure-preserving dynamical systems can be applied (see for instance [10]).

In contrast to Hamiltonian dynamics, however, the Hamiltonian form $\mathcal{K}$ is not necessarily preserved: using Cartan's magic formula [11], we have

$$\pounds_{X_P}\mathcal{K} = (\mathbf{d}\mathbf{i}_{X_P} + \mathbf{i}_{X_P}\mathbf{d})\mathcal{K} = \mathbf{d}\mathbf{i}_{X_P}\mathcal{K},$$

which is not necessarily zero.

Another difference with Hamiltonian dynamics is that the Hamiltonian form $\mathcal{K}$ is not uniquely determined by (10): whenever $\mathcal{K}$ is a solution of (10), then so is $\mathcal{K} + \mathbf{d}\phi$, with ϕ any $(n-3)$-form.

F. Example: Two-Dimensional Dynamics

We now revisit the system of Section II. We denote the coordinates on $\mathbb{R}^2$ by (p_x, p_y). The volume form Ω then becomes the symplectic form $dp_x \wedge dp_y$ and the gyroscopic form (6) is given by $F = F(p_x, p_y) dx \wedge dy$, where F is a function of the momenta alone. For the sake of simplicity, we assume furthermore that the mass matrix is given by $M = mI$, where I is the 2-by-2 identity matrix, so that the Hamiltonian becomes

$$H = \frac{1}{2m}(p_x^2 + p_y^2) - Ax - By.$$

The scalar condition (9) simplifies to

$$\frac{\partial F}{\partial p_x} p_y - \frac{\partial F}{\partial p_y} p_x = 0,$$

which implies that F depends on (p_x, p_y) through the distance to the origin:

$$F = f(p_x^2 + p_y^2),$$

for some function f. To find $\mathcal{K}$, we write (10) in local coordinates:

$$\frac{\partial \mathcal{K}}{\partial p_x} = -B - f(p_x^2 + p_y^2) p_x$$
$$\frac{\partial \mathcal{K}}{\partial p_x} = \quad A - f(p_x^2 + p_y^2) p_y,$$

which readily integrates to

$$\mathcal{K} = -\frac{1}{2} g(p_x^2 + p_y^2) + A p_y - B p_x,$$

where $g(t) = \int_0^t f(s) ds$. When $F = \lambda \left(p_x^2 + p_y^2\right)^{1/2}$ as in Section II-B, the function $\mathcal{K}$ coincides with the Hamiltonian obtained in Section II-C.

G. Example: Particle in a Constant Electromagnetic Field

We now consider the dynamics of a charged particle in $\mathbb{R}^3$ with charge e and mass m under the influence of an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. Both $\mathbf{E}$ and $\mathbf{B}$ are allowed to be time dependent but are supposed to be constant throughout space. The volume form Ω is the three-form $dp_x \wedge dp_y \wedge dp_z$.

The Hamiltonian is

$$H = \frac{1}{2m} \mathbf{p}^2 - e\mathbf{E} \cdot \mathbf{q}.$$

For the sake of convention, we will denote the gyroscopic two-form (6) in this case as $\mathfrak{B}$; it is given by

$$\mathfrak{B} = B_x dy \wedge dz + B_y dx \wedge dz + B_z dx \wedge dy.$$

As the components of $\mathfrak{B}$ are constant, the condition (13) is automatically satisfied. The Hamiltonian form $\mathcal{K}$ is now a one-form written in components as $\mathcal{K} = K_x dp_x + K_y dp_y + K_z dp_z$.

Using the Euclidian metric, we identify $\mathcal{K}$ with the vector $\mathbf{K}$ with components (K_x, K_y, K_z), and the exterior derivative $\mathbf{d}\mathcal{K}$ can then be identified with the curl $\nabla \times \mathbf{K}$. In Euclidian coordinates, the condition (10) then becomes

$$\nabla \times \mathbf{K} = e\mathbf{E} + m^{-1} \mathbf{p} \times \mathbf{B}, \tag{14}$$

which we recognize as the expression for the Lorentz force. The volume equations can now be written in Euclidian coordinates as $\dot{\mathbf{p}} = \nabla \times \mathbf{K}$, which results in the Lorentz equations in terms of $\mathbf{p}$.

If we view (14) as an equation for $\mathbf{K}$, we find that

$$\mathbf{K} = e\mathbf{E} \times \mathbf{p} + m^{-1} \|\mathbf{p}\|^2 \mathbf{B}.$$

Note that this equation determines $\mathbf{K}$ only up to a gradient vector field. This is the Euclidian analogue of the fact that the Hamiltonian form $\mathcal{K}$ is in general only determined up to an exact form, a fact pointed out at the end of Section III-E.

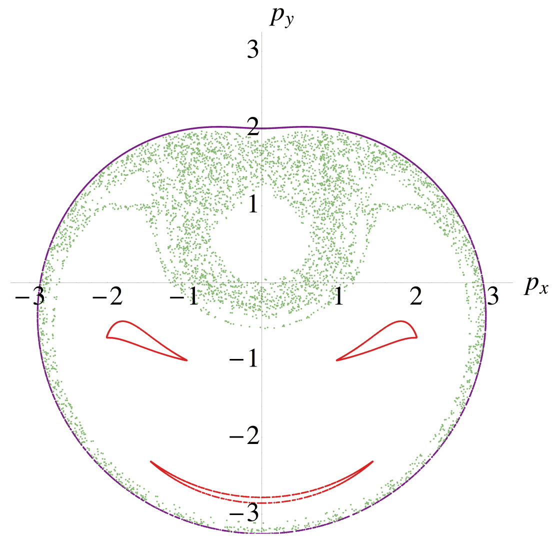

Fig. 2. Stroboscopic images of solutions to (5), with $\lambda = 1$ and $\phi(x, y) = -y \sin 2t$, corresponding to three different initial conditions. Each trajectory is represented by 5000 discrete points of a particular color; the set of points corresponding to the initial condition $(p_x, p_y)(0) = (0, 3/2)$ has approximate correlation dimension 1.8.

IV. CHAOTIC DYNAMICS

When λ or $\phi(x, y)$ varies explicitly with time in (5), complicated dynamics can result in the (p_x, p_y) plane. In terms of the metaphor of Fig. 1, the case in which λ varies periodically while $\phi(x, y)$ remains temporally constant corresponds to the situation in which a body navigates in a planar fluid against the backdrop of a stationary potential — like that of Earth's gravity — by periodically varying the position of an internal rotor. A variation on this system was examined in [4]. The remainder of the present paper considers the complementary case in which λ is constant but $\phi(x, y)$ varies periodically. This corresponds to the situation in which a body with constant spin — more generally, with a constant coefficient of lift — is subjected to a periodic external force.

We observe first that if λ is constant and $\nabla \phi = 0$ identically, then (5) may be solved explicitly. Solutions correspond to circular orbits around the origin in the (p_x, p_y) plane with period $2\pi / |\lambda| \sqrt{p_x(0)^2 + p_y(0)^2}$, executed clockwise if λ is positive and counterclockwise is λ is negative.

If $\nabla \phi$ is spatially constant but periodic in time, trajectories in the (p_x, p_y) plane may cross themselves. In order to visualize a trajectory resulting from periodic forcing with period T, we sample the trajectory at discrete times given by consecutive multiples of T to obtain a stroboscopic Poincaré section. Fig. 2 depicts three such sections, corresponding to different initial conditions, on a single set of axes.

The points comprising two of the sections in Fig. 2 lie along unions of closed curves, suggesting that the momentum of the body varies quasiperiodically with time. The points sampled from the trajectory beginning at $(p_x, p_y)(0) = (0, 3/2)$, however, belong to a more complicated set.

A Poincaré section exhibiting fractal geometry is a hall-

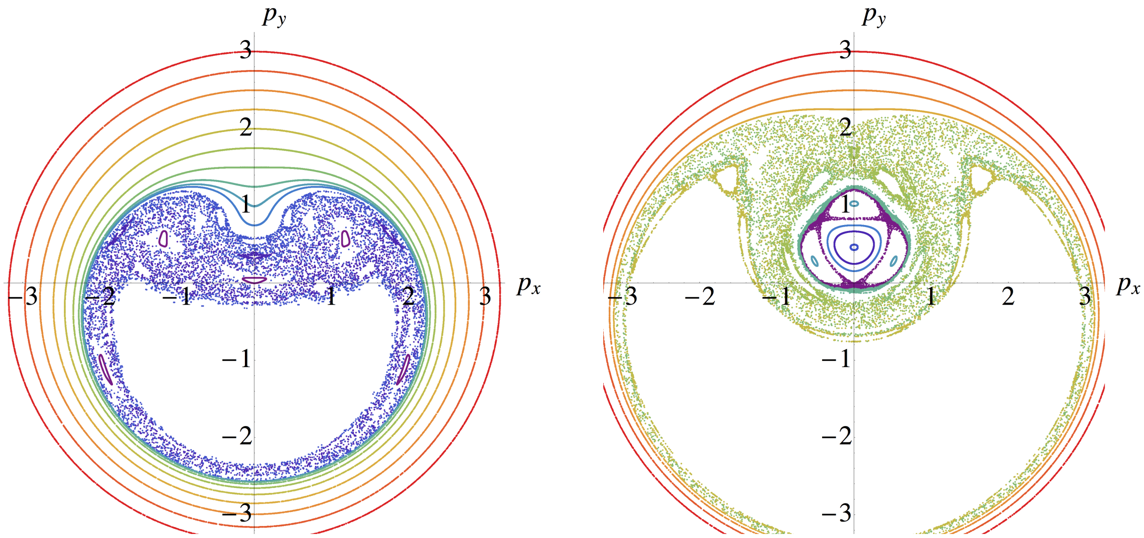

Fig. 3. Stroboscopic images of solutions to (5) with $\lambda = 1$ and $\phi = -y \sin \nu t$. The plot on the left corresponds to $\nu = 1.4$ and the plot on the right to $\nu = 2.2$. In each plot, a single color is assigned to the solution obtained from one of thirteen initial conditions spaced equally between $(p_x, p_y) = (0, 0)$ and $(p_x, p_y) = (0, 3)$. Each solution is sampled at time $t = 2k\pi/\nu$ for $\nu = 1, \ldots, 5000$. The trajectory corresponding to the initial condition $(p_x, p_y) = (0, 1/2)$ is represented in the left-hand plot by a diffuse collection of blue points with approximate correlation dimension 1.8. The trajectory corresponding to the initial condition $(p_x, p_y) = (0, 2)$ is represented in the right-hand plot by a diffuse collection of green points with approximate correlation dimension 1.2. The fractal geometry of each collection of points indicates chaotic dynamics.

mark of chaotic dynamics. When only a finite selection of points from a subset of the plane is available, the (possibly fractal) dimension of the complete subset may be estimated in a variety of ways [12]. For the system represented in Fig 2, we assess the geometry of the Poincaré section corresponding to the initial condition $(p_x, p_y)(0) = (0, 3/2)$ by computing the *correlation dimension* [13] of the set of 5000 points shown in green.

If ϵ is a real number greater than the minimum distance between green points in Fig. 2 and less than the maximum distance between such points, it's straightforward to compute

$$C(\epsilon) = \frac{\text{\# of pairs of points separated by less than } \epsilon}{\text{total \# of pairs of points}}.$$

Away from the extreme values allowed for ϵ, this quantity will vary with ϵ like

$$C(\epsilon) \sim \epsilon^D,$$

where D is the correlation dimension. For the set of green points in Fig. 2, the correlation dimension computed in this way is approximately 1.8.

Fig. 3 depicts Poincaré sections like those of Fig. 2 for two different forcing frequencies and a variety of initial conditions. In each case, the system is driven sinusoidally in the y direction by the potential force. Higher-resolution versions of these plots and plots corresponding to additional forcing frequencies are available at `http://kellyfish.net/cdc12/`.

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